



Irreducible matrices with reducible principal submatrices¹

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Received 13 July 1998; accepted 12 November 1998

Submitted by R.A. Brualdi

Abstract

Let A be a $(0, 1)$ -matrix of order $n \geq 3$ and let $s_i^0(A)$, $i = 1, \dots, n$, be the number of the off diagonal 0's in row and column i of A . We prove that if A is irreducible, and if all its principal submatrices of order $(n - 1)$ are reducible, then $s_i^0(A) \geq n - 1$, $i = 1, \dots, n$. This establishes the validity of a conjecture by B. Schwarz concerning strongly connected graphs and their primal subgraphs. © 1999 Elsevier Science Inc. All rights reserved.

AMS classification: 05C20; 05C50

Keywords: Irreducible matrix; Principal submatrix; Strongly connected digraph; Primal subgraph

1. Introduction and notation

Let $A = (a_{ij})$ be a $(0, 1)$ -matrix of order n and let I_n denote the set $(1, \dots, n)$. For given subsets α and β of I_n , let $A[\alpha, \beta]$ be the submatrix of A consisting of all rows numbered α_i , $\alpha_i \in \alpha$, and all columns numbered β_j , $\beta_j \in \beta$. A is said to be *reducible* if there exist nonempty subsets α, β of I_n such that

$$\alpha \cap \beta = \emptyset, \quad \alpha \cup \beta = I_n, \quad A[\alpha, \beta] = 0.$$

¹ The research was supported by the Fund for the Promotion of Research at the Technion.

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Otherwise, A is *irreducible*. Equivalently, A is reducible if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B and D are square matrices.

A is the adjacency matrix of a digraph (directed graph) G with n vertices and d arcs. A is irreducible if and only if its digraph is strongly connected [1]. A primal subgraph of G is the graph obtained from G by deleting one vertex, say i , and all the arcs going out from i or into it. The adjacency matrix corresponding to this primal subgraph is the principal submatrix A_i of A obtained by deleting row and column i of A .

In Ref. [4], B. Schwarz conjectured that if G is a strongly connected digraph without loops, with $n \geq 3$ vertices and d arcs, where $[n(n-1)/2] + 1 \leq d \leq n(n-1)$, then G has a strongly connected primal subgraph. He verified his conjecture for $n = 3, 4, 5$. The conjecture is a generalization of a result of Brualdi and Hwang ([2], Lemma 1.2(b)).

For a given $(0, 1)$ -matrix $A = (a_{ij})$ of order n , satisfying $a_{ii} = 0, i = 1, \dots, n$, we denote by $s_i^0(A)$ and $s_i^1(A), i = 1, \dots, n$, the number of the off diagonal 0's and 1's, respectively, in the union of row and column i of A , and by $s(A)$ we denote the total number of 1's in A . Obviously,

$$s_i^0(A) + s_i^1(A) = 2(n-1), \quad i = 1, \dots, n, \quad (1)$$

$$s(A) = \frac{1}{2} \sum_{i=1}^n s_i^1(A). \quad (2)$$

In terms of $(0, 1)$ -matrices, the conjecture of B. Schwarz reads as follows: Let $A = (a_{ij})$ be an irreducible $(0, 1)$ -matrix of order n , satisfying $a_{ii} = 0, i = 1, \dots, n$. If $s(A) \geq [n(n-1)/2] + 1$, then there exists at least one irreducible principal submatrix A_i of A .

In Section 2 we prove our main result: Let A be an irreducible $(0, 1)$ -matrix of order $n \geq 3$, satisfying $a_{ii} = 0, i = 1, \dots, n$. If all the principal submatrices of A are reducible, then $s_i^0(A) \geq n-1, i = 1, \dots, n$ (Theorem 1). This result implies the validity of the conjecture for all n (Corollary 2).

In Section 3 we consider the question of sharpness of the constant $[n(n-1)/2] + 1$.

2. Results

We now bring our theorem.

Theorem 1. *Let $A = (a_{ij})$ be an irreducible $(0, 1)$ -matrix of order $n \geq 3$, satisfying $a_{ii} = 0, i = 1, \dots, n$. If all the principal submatrices of order $(n-1)$, $A_i, i = 1, \dots, n$, of A are reducible, then*

$$s_i^0(A) \geq n-1, \quad i = 1, \dots, n. \quad (3a)$$

Proof. As $A_i, i = 1, \dots, n$, are reducible, there exist n pairs of nonempty subsets of $I_n, (k_i, \ell_i), i = 1, \dots, n$, such that

$$k_i \cap \ell_i = \emptyset, \quad k_i \cup \ell_i = I_n - (i), \quad A[k_i, \ell_i] = 0, \quad i = 1, \dots, n. \quad (4)$$

As A is irreducible, it follows that if $(\alpha_1, \dots, \alpha_p)$ is any subset of I_n , then

$$\left| \bigcup_{i=1}^p k_{\alpha_i} \right| + \left| \bigcap_{i=1}^p \ell_{\alpha_i} \right| < n \quad \text{if } \bigcap_{i=1}^p \ell_{\alpha_i} \neq \emptyset, \quad (5a)$$

$$\left| \bigcap_{i=1}^p k_{\alpha_i} \right| + \left| \bigcup_{i=1}^p \ell_{\alpha_i} \right| < n \quad \text{if } \bigcap_{i=1}^p k_{\alpha_i} \neq \emptyset. \quad (5b)$$

Here we use the notation $|\alpha|$ for the number of elements belonging to a given set α .

We continue our proof with two propositions.

Proposition 1. Let $1 < p \leq n$ and let $(\alpha_1, \dots, \alpha_p)$ be a subset of I_n such that

$$\left| \bigcup_{i=1}^p \ell_{\alpha_i} \right| < n. \quad (6)$$

Then there exists an integer $i_0, 1 \leq i_0 \leq p$, such that

$$\left| \bigcup_{\substack{i=1 \\ i \neq i_0}}^p \ell_{\alpha_i} \right| < \left| \bigcup_{i=1}^p \ell_{\alpha_i} \right|. \quad (7a)$$

The same result holds for k_{α_i} 's replacing the ℓ_{α_i} 's in Eqs. (6) and (7a).

Proof. It is enough to prove the proposition for the ℓ_{α_i} 's. Without loss of generality, we may assume that $(\alpha_1, \dots, \alpha_p) = (1, \dots, p)$. Let

$$\left| \bigcup_{i=1}^p \ell_i \right| = m_1 + m_2, \quad (8)$$

where m_1 is the number of elements of the set $\bigcup_{i=1}^p \ell_i$ belonging to $(1, \dots, p)$, and m_2 is the number of elements belonging to $(p+1, \dots, n)$. By Eqs. (6) and (8),

$$m_1 + m_2 < n. \quad (9)$$

As $n-p-m_2$ of the numbers $p+1, \dots, n$ do not belong to $\bigcup_{i=1}^p \ell_i$, they belong to $\bigcap_{i=1}^p k_i$. Hence,

$$\left| \bigcap_{i=1}^p k_i \right| \geq n - p - m_2. \quad (10a)$$

We now show that $m_1 < p$. Indeed, if $m_1 = p$, Eqs. (9) and (10a) imply

$$\left| \bigcap_{i=1}^p k_i \right| \geq n - m_1 - m_2 > 1. \quad (10b)$$

Hence, by Eqs. (5b) and (10b)

$$\left| \bigcup_{i=1}^p \ell_i \right| \leq m_1 + m_2 - 1. \quad (11)$$

Inequality (11) contradicts Eq. (8) and so $m_1 < p$.

As $m_1 < p$, a nonempty subset of $p - m_1$ of the numbers $1, \dots, p$ is disjoint with $\bigcup_{i=1}^p \ell_i$. We may assume that this subset is $(1, \dots, p - m_1)$. So, $(1, \dots, p - m_1) \subset k_i$, $i = p - m_1 + 1, \dots, p$. Also, as we observed above, $n - p - m_2$ of the numbers $p + 1, \dots, n$ belong to all k_i , $i = p - m_1 + 1, \dots, p$. Thus

$$\left| \bigcap_{i=p-m_1+1}^p k_i \right| \geq n - p - m_2 + p - m_1 = n - m_1 - m_2. \quad (12)$$

By Eqs. (5b), (9) and (12), applied to the set $(p - m_1 + 1, \dots, p)$, it follows that

$$\left| \bigcup_{i=p-m_1+1}^p \ell_i \right| \leq m_1 + m_2 - 1. \quad (13)$$

Hence, by Eqs. (8) and (13), there exists an integer i_1 such that $i_1 \in \bigcup_{i=1}^{p-m_1} \ell_i$ and $i_1 \notin \bigcup_{i=p-m_1+1}^p \ell_i$.

If $p - m_1 = 1$, set $i_0 = 1$.

If $p - m_1 > 1$, let α, β be integers such that $1 \leq \alpha < \beta \leq p - m_1$. As the numbers $1, \dots, p - m_1$ do not belong to $\bigcup_{i=1}^p \ell_i$, it follows that $\alpha \in k_\beta$ and $\beta \in k_\alpha$. Hence,

$$|k_\alpha \cup k_\beta| + |\ell_\alpha \cap \ell_\beta| = n. \quad (14)$$

From Eq. (5a) and Eq. (14) it follows that

$$\ell_\alpha \cap \ell_\beta = \emptyset.$$

That is, the sets ℓ_i , $i = 1, \dots, p - m_1$, are mutually disjoint. So, i_1 belongs to precisely one of the sets ℓ_i , $i = 1, \dots, p - m_1$. If $i_1 \in \ell_{i_0}$, $1 \leq i_0 \leq p - m_1$, then

$$\left| \bigcup_{\substack{i=1 \\ i \neq i_0}}^p \ell_i \right| < \left| \bigcup_{i=1}^p \ell_i \right|, \quad (7b)$$

and Proposition 1 is proved. \square

Remark 1. If condition (6) is not satisfied, then Proposition 1 is not true in general. Indeed, let $n = 5$ and let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A is irreducible and all $A_i, i = 1, \dots, n$, are reducible. Here the five pairs of subsets of $I_5, (k_i, \ell_i)$, are given by

$$(k_i, \ell_i) = ((i-1), (i+1, i+2, i+3)), \quad i = 1, \dots, 5,$$

where the elements of k_i and ℓ_i are taken modulo 5 (and so belong to I_5). As

$$\left| \bigcup_{i=1}^4 \ell_i \right| = |I_5| = 5 = \left| \bigcup_{\substack{i=1 \\ i \neq i_0}}^4 \ell_i \right|,$$

where i_0 is any integer belonging to I_4 , it follows that Proposition 1 does not hold in this case.

Proposition 2. Let $1 \leq p \leq n$ and let $(\alpha_1, \dots, \alpha_p)$ be a subset of I_n . Then

$$\left| \bigcup_{i=1}^p k_{\alpha_i} \right| \geq p, \quad (15a)$$

$$\left| \bigcup_{i=1}^p \ell_{\alpha_i} \right| \geq p. \quad (15b)$$

Proof. It is enough to prove Eq. (15b), and without loss of generality we may assume that $(\alpha_1, \dots, \alpha_p) = (1, \dots, p)$; i.e., we will prove that

$$\left| \bigcup_{i=1}^p \ell_i \right| \geq p. \quad (15c)$$

For $p = 1$, Eq. (15c) holds, as ℓ_1 is a nonempty set.

If Eq. (15c) does not hold for $p = 2$, then $\ell_1 = \ell_2 = (j)$, where $j \in I_n, j \neq 1, 2$, and $k_1 = I_n - (1, j), k_2 = I_n - (2, j)$. As

$$|k_1 \cup k_2| = |I_n - (j)| = n - 1, \quad |\ell_1 \cap \ell_2| = |(j)| = 1, \quad (16)$$

Eq. (16) contradicts Eq. (5a), and so Eq. (15c) holds for $p = 2$.

Assume that Eq. (15c), hence Eq. (15b), does not hold for some $p, 3 \leq p \leq n$. Let $q, 3 \leq q \leq n$, be the smallest integer for which Eq. (15b) is not true. That is,

$$\left| \bigcup_{i=1}^q \ell_i \right| < q, \quad (17)$$

and Eq. (15b) holds for all $1 \leq p \leq q-1$. By Eq. (17),

$$\left| \bigcup_{i=1}^q \ell_i \right| < n.$$

Hence, by Proposition 1, there exists an integer $i_0, 1 \leq i_0 \leq q$, such that

$$\left| \bigcup_{\substack{i=1 \\ i \neq i_0}}^q \ell_i \right| < \left| \bigcup_{i=1}^q \ell_i \right|. \quad (18)$$

Inequalities (17) and (18) imply

$$\left| \bigcup_{\substack{i=1 \\ i \neq i_0}}^q \ell_i \right| < q-1. \quad (19)$$

Inequality (19) contradicts the choice of q as the smallest integer for which Eq. (15b) does not hold. Proposition 2 is thus proved. \square

Remark 2. By the theorem of Hall ([1], p. 7), Proposition 2 means that each of the systems of subsets k_1, \dots, k_n and ℓ_1, \dots, ℓ_n has a system of distinct representatives.

We now use Proposition 2 to complete the proof of Theorem 1.

For a given $j, 1 \leq j \leq n$, divide the set $I_n - (j)$ into two disjoint subsets defined by

$$I'_{n,j} = \{i | j \in k_i\}, \quad I''_{n,j} = \{i | j \in \ell_i\}.$$

Let

$$|I'_{n,j}| = n'_j, \quad |I''_{n,j}| = n''_j. \quad (20)$$

Then

$$n'_j + n''_j = n-1. \quad (21)$$

By Eq. (4),

$$A \left[(j), \bigcup_{i \in I'_{n,j}} \ell_i \right] = 0, \quad (22)$$

$$A \left[\bigcup_{i \in I''_{n,j}} k_i, (j) \right] = 0. \quad (23)$$

Hence,

$$s_j^0(A) \geq \left| \bigcup_{i \in I'_{n,j}} \ell_i \right| + \left| \bigcup_{i \in I''_{n,j}} k_i \right|. \quad (24)$$

By Eqs. (15a), (15b), (20), (21) and (24),

$$s_j^0(A) \geq n - 1. \quad (3b)$$

This completes the proof of Theorem 1. \square

Let A be a $(0, 1)$ -matrix satisfying the assumptions of Theorem 1. Then Eqs. (1)–(3a) imply

$$s_i^1(A) \leq n - 1, \quad i = 1, \dots, n, \quad (25)$$

$$s(A) \leq \frac{n(n-1)}{2}. \quad (26)$$

Consider the digraph G corresponding to A . As $s_i^1(A)$ is the number of arcs going out from vertex i of G or into it, and $s(A)$ is the total number of arcs of G , inequalities (25) and (26) imply the following corollaries.

Corollary 1. *Let G be a strongly connected digraph without loops and with n vertices such that none of its primal subgraphs is strongly connected. Then for each vertex i of G , the number of arcs going out from i or into it is at most $n - 1$ (i.e., the degree, or valency, of each vertex of G is at most $n - 1$).*

Corollary 2. *Let G be a strongly connected digraph without loops, with $n \geq 3$ vertices and d arcs and let $d_n = \lceil n(n-1)/2 \rceil + 1$. If $d \geq d_n$, then G has a strongly connected primal subgraph.*

Corollary 2 confirms the conjecture of B. Schwarz for all $n \geq 3$.

3. Sharpness of the constant $d_n = \lceil n(n-1)/2 \rceil + 1$

In this section we investigate the sharpness of the constant $d_n = \lceil n(n-1)/2 \rceil + 1$ in Corollary 2.

The constant is sharp for a given n if there exists an irreducible $(0, 1)$ -matrix of order n , $A = (a_{ij})$, $a_{ii} = 0$, $i = 1, \dots, n$, such that $s(A) = n(n-1)/2$ and all

the principal submatrices of A of order $(n-1)$ are reducible. If such a matrix exists for a given n , we will call it an *extremal matrix of order n* .

Let A be an extremal matrix of order n . We will derive some necessary conditions that A satisfies.

As $s(A) = n(n-1)/2$, it follows from Eqs. (2) and (25) that

$$s_i^1(A) = n-1, \quad i = 1, \dots, n. \quad (27)$$

That is, the degree of each vertex of a digraph G corresponding to an extremal matrix of order n is $n-1$ (i.e., G is regular of degree $n-1$).

From Eqs. (1) and (27) it follows that

$$s_i^0(A) = n-1, \quad i = 1, \dots, n. \quad (28)$$

Moreover, by Eqs. (3b), (15a), (15b), (20), (21) and (24), if Eq. (28) holds for a given j , $1 \leq j \leq n$, then

$$\left| \bigcup_{i \in I'_{n,j}} \ell_i \right| = |I'_{n,j}| = n'_j, \quad \left| \bigcup_{i \in I''_{n,j}} k_i \right| = |I''_{n,j}| = n''_j. \quad (29)$$

Hence, using Eqs. (22), (23) and (29), it follows that the number of 0's belonging to row (column) j , $j = 1, \dots, n$, of an **extremal matrix**, is $n'_j(n''_j)$; that is, it is equal to the number of pairs (k_i, ℓ_i) , $i \neq j$, for which j belongs to the k_i 's (ℓ_i 's).

Also, from Eq. (29) and Proposition 1, it follows that the sets $\ell_i(k_i)$, $i \in I'_{n,j}$ ($i \in I''_{n,j}$), can be arranged so that the number of elements belonging to the union of the first p , $1 \leq p \leq n'_j$ ($1 \leq p \leq n''_j$), ℓ_i 's (k_i 's) in that order, is equal to p . It follows that the first of the ℓ_i 's (k_i 's) in that order is a one element set, and the corresponding k_i (ℓ_i) is thus an $(n-2)$ element set. So every j , $1 \leq j \leq n$, belongs to some set k_i (ℓ_i) so that $|k_i| = n-2$ ($|\ell_i| = n-2$). This implies that the set of pairs (k_i, ℓ_i) , $i = 1, \dots, n$, corresponding to an extremal matrix, contains at least two pairs with one element k 's and at least two pairs with one element ℓ 's.

In Ref. [4], B. Schwarz showed that $d_3 = 4$ and $d_4 = 7$ are sharp, and he left the question of the sharpness of $d_5 = 11$ open.

An extremal matrix for $n = 3$ is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here the pairs (k_i, ℓ_i) , $i = 1, 2, 3$, are $((3), (2))$, $((1), (3))$, $((2), (1))$.

An extremal matrix for $n = 4$ is [4]

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Here the pairs $(k_i, \ell_i), i = 1, \dots, 4$, are $((3,4),(2)), ((1),(3,4)), ((1,2),(4)), ((3), (1,2))$.

Using the necessary conditions for extremal matrices, we will show that an extremal matrix of order five does not exist.

For a given pair (k_i, ℓ_i) , we call $(|k_i|, |\ell_i|)$ the pattern of that pair. If a number belongs to $k_i(\ell_i)$, we will say that it belongs to the rows (columns) of the pair (k_i, ℓ_i) .

Assume that there exists an extremal matrix A of order five and consider the corresponding pairs $(k_i, \ell_i), i = 1, \dots, 5$. By the necessary conditions, two of the pairs are of pattern (1,3), two of pattern (3,1) and the fifth is necessarily of pattern (2,2) (otherwise, the row sum of A cannot be equal to the column sum). As the elements belonging to the one element sets of the pairs have to be distinct, we may choose them to be the numbers 1–4. Thus the five pairs, denoted (I)–(V), are:

(I) $((\cdot, \cdot, \cdot), (1)),$

(II) $((\cdot, \cdot, \cdot), (2)),$

(III) $((3), (\cdot, \cdot, \cdot)),$

(IV) $((4), (\cdot, \cdot, \cdot)),$

(V) $((\cdot, \cdot), (\cdot, \cdot)),$

where the dots stand for numbers to be placed. Accordingly, the number of (off diagonal) zeros in row (column) i of A is

i	No. of 0's in row i	No. of 0's in column i
1	1	3
2	1	3
3	3	1
4	3	1
5	2	2

We will distinguish between the following two cases.

Case 1. The number 5 does not belong to (V). The number 5 thus belongs to all four pairs (I)–(IV). As there is only one zero in row 1, the number 1 does not belong to the rows in (V), and so it does belong to the columns in (V). This, and similar arguments regarding the numbers 2, 3 and 4, determines (V) to be

(V) $((3,4), (1,2)).$

As there are three zeros in column 1, it follows from (V) that the numbers 3 and 4 belong to the rows of (I), and so (I) is determined to be

(I) $((3, 4, 5), (1))$.

Similarly, (II) is determined to be

(II) $((3, 4, 5), (2))$.

(I) and (II) together mean that A is reducible.

Case 2. The number 5 belongs to (V). A straightforward lengthy similar discussion, which I will not elaborate, leads to a contradiction in this case too.

We proved that an extremal matrix of order 5 does not exist, and so $d_5 = 11$ in Corollary 2 is not sharp.

The matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

is irreducible, all its principal submatrices are reducible and $S(A) = 9$. This shows that the constant $d_5 = 11$ cannot be decreased below 10. Hence, for $n = 5$, Corollary 2 holds for $10 \leq d$, and 10 is sharp.

For $n \geq 6$, the question whether the constant $d_n = \lfloor n(n-1)/2 \rfloor + 1$ is sharp or not remains open.

Remark 3. Let $A = (a_{ij})$ be an irreducible $(0,1)$ tournament matrix of order $n \geq 4$. As A has an irreducible principal submatrix of order $n-1$ ([3], Theorem 3), it follows that an extremal matrix of order n cannot be a tournament matrix. This reduces the number of “candidates” for extremal matrices.

Acknowledgements

The author would like to thank Professor B. Schwarz for drawing his attention to his conjecture and for his valuable remarks. Thanks also to the referee for his comments.

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